1. *Prove that the Principle of Mathematical Induction, the Principle of Strong Induction and the well ordering property of $\mathbb{N}$ are all equivalent.*

   Solution: The following are equivalent:
   1)Principle of Mathematical Induction
   2)Principle of Strong Induction
   3)Well Ordering property of $\mathbb{N}$

   Proof:
   $(2) \implies (3)$**: Let S be a non-empty subset of $\mathbb{N}$. Suppose S has no minimal element. $n = 1$ does not belong to S as it will be the minimal element. For the same reason $n = 2$ does not belong to S as $n = 1 \notin S$.Suppose $\{1, \ldots, n\} \notin S$, then $n + 1 \notin S$ as then it will be the minimal element. So, by Principle of Strong Induction S should be empty , which is a contradiction.**
   $(3) \implies (1)$**:Let $P(n)$ be a mathematical statement for $n \in \mathbb{N}$.Suppose $P(1)$ is true and $P(n+1)$ is true whenever $P(n)$ is true. If $P(k)$ is not true for all integers,then S be the non-empty set of k for which $P(k)$ is not true. By Well Ordering property it should have a minimal element which cannot be $k = 1$. Hence $P(k-1)$ is true which implies that $P(k)$ is true,which is a contradiction.**
   $(2) \implies (1)$**: it is trivial.**
   $(1) \implies (2)$**: Let us assume that for the statement $P(n)$ the hypothesis of Strong induction are met and let $Q(k)$ be the statement that '$P(n)$ is true for all $n \leq k$'**
   We need to show that $Q(n)$ is true for all $n \in \mathbb{N}$ using the conditions of Regular Induction.
   Now let us check the conditions for Regular Induction:
   Since $P(1)$ is true $Q(1)$ is true.
   Since we have assumed the conditions of Strong Induction '$P(n)$ is true for all $n \leq k$' implies $P(k+1)$ is true, i.e. $Q(k)$ is true implies $P(k+1)$ is true.
   But,$Q(k)$ is true and $P(k+1)$ is true together implies $Q(k+1)$ is true.Hence, we get that $Q(k)$ is true implies $Q(k+1)$.
   Thus, by Regular Induction we get that $Q(n)$ is true for all $n \in \mathbb{N}$, which in turn says that $P(n)$ is true for all $n \in \mathbb{N}$. □

2. *Prove that if A_n is a countable set for each $n \in \mathbb{N}$, then their union $\bigcup_{n=1}^{\infty} A_n$ is countable. Prove that if $B_1 \ldots B_r$ are a finite number of countable sets then their product $\Pi_{n=1}^{r} B_n$ is countable.*

   Solution:Let us first prove that if $B_1 \ldots B_r$ are a finite number of countable sets then their product $\Pi_{n=1}^{r} B_n$ is countable.
   It's sufficient to prove for the case for $r = 2$, as for the general case can be done by induciton.

   We will try to set a bijection from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$
   Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by:

$$f(m, n) = 2^m 3^n$$

if $f(m,n) = f(l,k)$, by fundamental theorem of arithmetics we get that $m = l, n = k$
We can also define an injection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ say by $g(n) = (0, n)$
Hence, using the Cantor-Berstein theorem we can say that there exists a bijection $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
Thus we get that if B_1andB_2 are countable then $B_1 \times B_2$ is injectively mapped into a subset of $\mathbb{N} \times \mathbb{N}$ and since the later is countable , any subset of it is also so, hence $B_1 \times B_2$ is also countable.

For the union case:
Since A_n is countable for each $n \in \mathbb{N}$ we can write $A_n = \{a_{1,n}, a_{2,n}, a_{3,n} \ldots\}$.
Let $A = \bigcup_{n=1}^{\infty} A_n$
Case 1: Suppose A_n is a collection of disjoint sets. Let $x \in A$, then $x \in A_k$ for some k and hence $x = (m,n)$ for some $m, n \in \mathbb{N}$.The pair (m,n) is uniquely determined as the sets are disjoint.
Let the function $f : A \to \mathbb{N} \times \mathbb{N}$ be defined by $f(x) = (m,n)$ Then f is an injection into a subset of $\mathbb{N} \times \mathbb{N}$ and hence countable.

and since we have the following lemma:

Lemma 0.1 ***If** $F = \{A_1, \ldots, A_n, \ldots,\}$ **is a collection of countable sets. Let** $G = \{B_1, \ldots, B_n, \ldots,\}$ **be such that** $B_1 = A_1$ **and** $B_n = A_n - \bigcup_{k=1}^{n-1} A_n$ **for** $n > 1$.**Then** $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$*

we get that $\bigcup_{n=1}^{\infty} A_n$ is countable.

□

3. *State the completeness property of $\mathbb{R}$. Using, the completeness property of $\mathbb{R}$, prove that there exists a positive real number x such that $x^2 = 5$. Prove that the ordered field $\mathbb{Q}$ of rational numbers does not satisfy the completeness property.*

   Solution:Statement: Every non-empty set S of real numbers which is bounded above has a supremum; that is there is a real number b such that sup $S = b$.

   If we consider the set S= $\{x \in \mathbb{R} : x^2 \leq 5\}$, it should have a supremum as it is a bounded set, i.e. there exists a positive real number x such that $x^2 = 5$.

   $\mathbb{Q}$ does not satisfy the Completeness property because the subset $\{r \in \mathbb{Q} : r^2 < 5\}$ is bounded but do not have a supremum in $\mathbb{Q}$. □

4. *Prove that a cauchy sequence of real numbers converges.*

   Solution:Let $\{x_n\}$ be the given cauchy sequence of real numbers, which means given $\epsilon > 0 \ \exists \ k \in \mathbb{N}$ such that:

   $$\mid x_n - x_m \mid < \epsilon/2, \ \ \forall n, m \geq k$$

   which implies:

   $$\mid x_n - x_k \mid < \mid x_n - x_{k+1} \mid + \mid x_{k+1} - x_k \mid < \epsilon, \ \ \forall n \geq k$$

   let us choose $\epsilon = 1$ for our convenience.
   thus, we get that $x_n < 1 + x_k \ \ \forall n \geq k$

if we set $M = max\{x_1, \ldots, x_{n-1}, 1 + x_k\}$, then we get that $\{x_n\}$ is bounded by M.

By Boltzano-Weirstrass theorem we get that every bounded infinite set has a limit point.
So, $\{x_n\}$ has a limit point , i.e. $\{x_n\}$ is a convergent sequence. $\square$

5. *Compute $lim_{n\to\infty} n^{\frac{1}{n}}$*

   Solution: Using, $A.M. \geq G.M.$ we get

$$\frac{1 + 1 + \ldots + \sqrt{n} + \sqrt{n}}{n} \geq n^{\frac{1}{n}} > 1$$

   which implies

$$1 - \frac{2}{n} + \frac{2}{\sqrt{n}} \geq n^{\frac{1}{n}} > 1$$

   which shows $lim_{n\to\infty} n^{\frac{1}{n}} = 1$ $\square$

6. *Let $X = \{x_n\}$ be a sequence of real numbers which converges.Let $limx_n = x$, where $x \in \mathbb{R}$. Let $Y = \{y_n\}$ be another sequence of real numbers.*
   Prove that $limsup_{n\to\infty}(x_n + y_n) = x + limsup(y_n)$.

   Solution: let us assume $x = 0$, and $l = limsup(y_n)$, then given $\epsilon > 0$ we get a $N \in \mathbb{N}$ such that

$$y_n < l + \frac{\epsilon}{2}, \quad \forall n > N$$

   and, given $m > 0$ there esists an integer $n > m$ such that

$$y_n > l - \frac{\epsilon}{2},$$

   since $limx_n = 0$, given $\epsilon > 0$ we get a $k \in \mathbb{N}$ such that

$$\mid x_n \mid < \frac{\epsilon}{2} \quad \forall n \geq k$$

   if we set $h = max\{N, k\}$, we get that

$$x_n + y_n < l + \epsilon, \quad \forall n > h$$

   and given $m > 0$ there exists a $n > h > m$ such that

$$x_n + y_n > l - \epsilon,$$

   hence, we get that $limsup(x_n + y_n) = limsup(y_n)$
   Thus, if $\lim x_n = x$ then $lim(x_n - x) = 0$ and hence $limsup(x_n - x + y_n) = limsup(y_n)$ which implies that $limsup_{n\to\infty}(x_n + y_n) = x + limsup(y_n)$. $\square$

7. *Show that the sequence of real numbers $x_n = \sum_{i=1}^{n}(-1)^{i+1}/i$ converges*

   Solution:if $m > n \geq N$, we find that

$$\mid x_m - x_n \mid = \mid \frac{1}{n+1} - \frac{1}{n+2} + \ldots \pm \frac{1}{m} \mid < \frac{1}{n} \leq \frac{1}{N}$$

   By Archimedian property given $\epsilon > 0\ \exists\ k \in \mathbb{N}$ such that $0 \leq \frac{1}{N} < \epsilon\ \forall\ N \geq k$.
   Thus we get that $\{x_n\}$ is a cauchy sequence and hence convergent. $\square$